

A Finer Classification of Cellular Automata

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Background

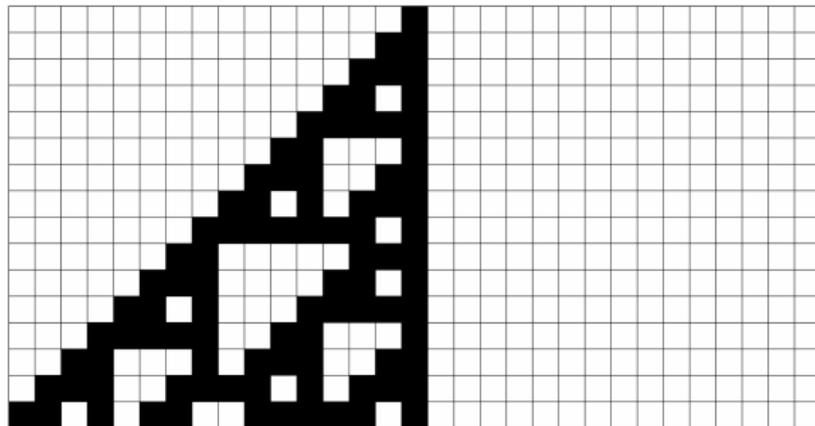
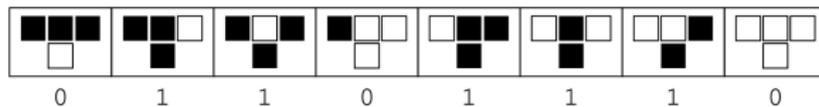
1. Early research included John von Neumann's universal constructor for self-replication. It's also computationally universal.
2. The specification of the universal constructor is quite complex. Some suspected its complexity is due to complicated rules, but Conway's Game of Life provided a CA with simple rules that's Turing-complete.
3. Wolfram provided a classification of cellular automata based on his own observations.

Definitions

1. Let \mathbb{Z}^n be the n -dimensional grid. For $x \in \mathbb{Z}^n$, we call $x + \{-1, 0, 1\}^n$ the 1-neighborhood of x .
2. A function $c : \mathbb{Z}^n \rightarrow \{0, 1\}$ or $c : x + \{-1, 0, 1\}^n \rightarrow \{0, 1\}$ is called a 2-coloring.
3. A rule is a function $R : \{\{-1, 0, 1\}^n \rightarrow \{0, 1\}\} \rightarrow \{0, 1\}$.
4. For each $c : x + \{-1, 0, 1\}^n \rightarrow \{0, 1\}$, let $c_x : \{-1, 0, 1\}^n \rightarrow \{0, 1\}$ be the map with origin at x . A n -dimensional 2-color 1-neighbor cellular automaton, or simply a n -dimensional CA, is a map $F : \{\mathbb{Z}^n \rightarrow \{0, 1\}\} \rightarrow \{\mathbb{Z}^n \rightarrow \{0, 1\}\}$ such that $F(c)(x) = R(c_x)$ for some rule R .

Example: Rule 110, a 1-dimensional CA

rule 110



Example: Game of Life, a 2-dimensional CA

1. Any live cell with fewer than two live neighbours dies, as if by underpopulation.
2. Any live cell with two or three live neighbours lives on to the next generation.
3. Any live cell with more than three live neighbours dies, as if by overpopulation.
4. Any dead cell with exactly three live neighbours becomes a live cell, as if by reproduction.

Wolfram Class 4

Wolfram Class 4 refers to a collection of intricate CAs that are Turing-complete. Matthew Cook (2004) proved Rule 110 is Turing-complete. Another CA, Conway's Game of Life is also proved Turing-complete by Conway himself (1982). In the following slides, we are going to show that Wolfram Class 4 isn't fine enough to tell the difference of a host of interesting properties, like reversibility, oracles, and chirality.

Reversible CA

1. Richardson (1972) proved that if a CA is reversible, then its inverse is also a CA.
2. Moore (1962) and Myhill (1963) proved that injectivity, bijectivity, and reversibility are equivalent for CAs.
3. Morita and Harao (1989) proved that there exists 1-dimensional reversible CA that's Turing-complete. Since Rule 110 is irreversible and Turing-complete, Wolfram Class 4 tells no difference between reversibility.
4. Proposition: There exists a \mathbb{Z} group action that's Turing-complete.

Oracles

1. A oracle is a machine that can answer whether a element is in a set.
2. CAs with inputs consisting of finite 0s or 1s are TM (Turing Machine) computable.
3. Due to wave nature of CAs, CAs with inputs consisting of all 2-colorings are OTM (Oracle Turing Machine) computable with the input as oracle.
4. Both Rule 110 and Game of Life are Turing-complete whether the input space is finite 0s or 1s, or all 2-colorings. Knowing Turing-completeness doesn't fully characterize the computational properties of CAs.

Chirality

1. A rule R is called i -non-chiral if R is invariant under $1 \leftrightarrow -1$ in the i -th coordinate. Otherwise it's called i -chiral. A CA is i -chiral/ i -non-chiral if the rule is.
2. A rule R is called non-chiral if R is i -non-chiral for all i . Otherwise it's called chiral. A CA is chiral/non-chiral if the rule is.
3. Problem: Is Wolfram Class 4 fine enough to tell the difference in chirality?
4. Result: For dimension $n \geq 2$, the answer is no. For $n = 1$, it's a open problem.

Embedding

1. Let $n > m$, a 2-coloring $c : \{-1, 0, 1\}^n \rightarrow \{0, 1\}$ can be restricted to the first m coordinates $c : \{-1, 0, 1\}^m \times \{0\}^{n-m} \rightarrow \{0, 1\}$ and canonically identified with another 2-coloring $c' : \{-1, 0, 1\}^m \rightarrow \{0, 1\}$.
2. A m -dimensional rule R_1 is said to be embedded in a n -dimensional rule R_2 if $R_2(c) = R_1(c')$.
3. Proposition: Let $n > m$. Every m -dimensional rule admits an embedding into a n -dimensional rule.
4. Corollary: If the m -dimensional rule is Turing-complete, so is the n -dimensional rule obtained by embedding.

Reversible/Irreversible Embedding

1. Proposition: Let $n > m$. Given a m -dimensional reversible/irreversible rule R , R admits a embedding into a n -dimensional reversible/irreversible rule.
2. In other words, embedding preserves reversibility and irreversibility.

Chiral/Non-Chiral Embedding

1. Proposition: Let $n > m$. Given a m -dimensional i -chiral/ i -non-chiral rule R , R admits a embedding into a n -dimensional i -chiral/ i -non-chiral rule.
2. Proposition: Let $n > m$. Given a m -dimensional chiral/non-chiral rule R , R admits a embedding into a n -dimensional chiral/non-chiral rule.
3. In other words, embedding preserves chirality.

Main Result

1. For all $n \geq 1$, there exists a n -dimensional reversible rule that is Turing-complete.
2. For all $n \geq 1$, there exists a n -dimensional chiral rule that is Turing-complete.
3. For all $n \geq 2$, there exists a n -dimensional irreversible rule that is Turing-complete.
4. For all $n \geq 2$, there exists a n -dimensional non-chiral rule that is Turing-complete.
5. Wolfram Class 4 tells no difference in reversibility and chirality for $n \geq 2$.
6. Open Problem: For $n = 1$, it's not known whether non-chiral Turing-complete rules exist.

Examples

1. Rule 110 is a 1-dimensional irreversible chiral rule. For all $n \geq 1$, there exists a n -dimensional irreversible chiral rule that is Turing-complete.
2. Game of Life is a 2-dimensional irreversible non-chiral rule. For all $n \geq 2$, there exists a n -dimensional irreversible non-chiral rule that is Turing-complete.

A Finer Classification of Cellular Automata

1. We can classify CAs according to whether they are Turing-complete, reversible, computable by TM/OTM, and chiral/non-chiral, whereas Wolfram Class 4 lumps CAs that are Turing-complete together.
2. The finer classification is useful if geometry is a prominent concern, like in physics. While electromagnetism is a non-chiral theory, electroweak interactions are chiral. Therefore, telling chirality can rule out non-chiral CAs as models of electroweak interactions.