A Geometrical Study of Cellular Automata

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1 Introduction

Early study of cellular automata is illustrated by John von Neumann's universal constructor [4] that models self-replication. The behavior is quite complex. As some might suspect the complexity of such patterns comes from the complexity of rules, but John Horton Conway's Game of Life only utilizes simple rules, and yet produces complex patterns that are Turing-complete. The possibility of complexity arising from simple rules is then a fascinating area for research.

In *A New Kind of Science* [6], Stephen Wolfram systematically discussed cellular automata. Based on his observations, cellular automata are classified into four classes, Class 1 that results in repetition, Class 2 that results in nested structures, Class 3 that results in randomness, and Class 4 that contains localized structures. Class 4 is of special interest and Wolfram speculated it's Turing-complete [5]. Rule 110 is Class 4 and proved [1] to be Turing-complete.

Since then, there is a lot of research focused on computational properties of cellular automata. Although there are works that demonstrate dimension may affect computation [2], and elucidate the relationship between symmetry and algebraic properties [3], the study of geometric properties in cellular automata remains remote. It's the purpose of this manuscript to take a first step to fill that gap.

1.1 Early Research

1.2 Modern Research

2 A Geometrical Study of Cellular Automata

2.1 Cellular Automata

Definition 1 *Let $\mathbb{Z}^n$ be the n-dimensional grid. For $x \in \mathbb{Z}^n$, we call $x + \{-1,0,1\}^n$ the* 1*-neighborhood of x.*

Definition 2 *A function $c : \mathbb{Z}^n \to \{0,1\}$ or $c : x + \{-1,0,1\}^n \to \{0,1\}$ is called a 2-coloring.*

Definition 3 *A rule is a function $R : \{\{-1,0,1\}^n \to \{0,1\}\} \to \{0,1\}$.*

Definition 4 *Given $c : \mathbb{Z}^n \to \{0,1\}$, for each $c|_x : x + \{-1,0,1\}^n \to \{0,1\}$, let $c_x : \{-1,0,1\}^n \to \{0,1\}$ be the map with origin at x. A n-dimensional* 2*-color* 1*-neighbor cellular automaton, or simply a n-dimensional CA, is a map*

$F : \{\mathbb{Z}^n \to \{0,1\}\} \to \{\mathbb{Z}^n \to \{0,1\}\}$ such that $F(c)(x) = R(c_x)$ for some rule R.

2.2 Embedding

Definition 5 *Let $n > m$, a 2-coloring $c : \{-1,0,1\}^n \to \{0,1\}$ can be restricted to the first m coordinates $c : \{-1,0,1\}^m \times \{0\}^{n-m} \to \{0,1\}$ and canonically identified with another 2-coloring $c' : \{-1,0,1\}^m \to \{0,1\}$.*

A m-dimensional rule R_1 is said to be embedded in a n-dimensional rule R_2 if $R_2(c) = R_1(c')$.

Proposition 6 *Let $n > m$. Every m-dimensional rule admits a embedding into a n-dimensional rule.*

Corollary 7 *If the m-dimensional rule is Turing-complete, so is the n-dimensional rule obtained by embedding.*

2.3 Reversibility

Proposition 8 *There exists a $\mathbb{Z}$ group action that's Turing-complete.*

Proposition 9 *Let $n > m$. Given a m-dimensional reversible/irreversible rule R, R admits a embedding into a n-dimensional reversible/irreversible rule.*

In other words, embedding preserves reversibility and irreversibility.

2.4 Chirality

Definition 10 *A rule R is called i-non-chiral if R is invariant under $1 \leftrightarrow -1$ in the i-th coordinate. Otherwise it's called i-chiral. A CA is i-chiral/i-non-chiral if the rule is.*

Definition 11 *A rule R is called non-chiral if R is i-non-chiral for all i. Otherwise it's called chiral. A CA is chiral/non-chiral if the rule is.*

Proposition 12 *Let $n > m$. Given a m-dimensional i-chiral/i-non-chiral rule R, R admits a embedding into a n-dimensional i-chiral/i-non-chiral rule.*

Proposition 13 *Let $n > m$. Given a m-dimensional chiral/non-chiral rule R, R admits a embedding into a n-dimensional chiral/non-chiral rule.*

In other words, embedding preserves chirality.

2.5 Main Results

Proposition 14 *Rule 110 is a 1-dimensional irreversible chiral rule. For all $n \geq 1$, there exists a n-dimensional irreversible chiral rule that is Turing-complete.*

Proposition 15 *Game of Life is a 2-dimensional irreversible non-chiral rule. For all $n \geq 2$, there exists a n-dimensional irreversible non-chiral rule that is Turing-complete.*

References

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